Rings and modules which are stable under automorphisms of their injective hulls

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Abstract

It is proved, among other results, that a prime right nonsingular ring (in particular, a simple ring) R is right self-injective if R_R is invariant under automorphisms of its injective hull. This answers two questions raised by Singh and Srivastava, and Clark and Huynh. An example is given to show that this conclusion no longer holds when prime ring is replaced by semiprime ring in the above assumption. Also shown is that automorphism-invariant modules are precisely pseudo-injective modules, answering a recent question of Lee and Zhou. Furthermore, rings whose cyclic modules are automorphism-invariant are investigated.

1. Introduction and Preliminaries

Throughout, R will denote an associative ring with identity and modules will be right modules. In [4] Dickson and Fuller studied modules which are invariant under automorphisms of their injective hulls, when the underlying ring is a finite dimensional algebra over a field with more than two elements. Such modules over arbitrary rings were discussed by Lee and Zhou in [9], where they were called automorphism-invariant modules. Thus, a module M is called an automorphism-invariant module if M is invariant under any automorphism of its injective hull. Clearly every (quasi-)injective module is automorphism-invariant.

Dickson and Fuller had shown that if R is a finite-dimensional algebra over a field with more than two elements, then R is of right invariant module type if and only if every indecomposable right R-module is automorphism-invariant. Recently, Singh and Srivastava have investigated in [12] rings whose finitely generated indecomposable right modules are automorphism-invariant, and completely characterized indecomposable right Artinian rings with this property.

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The dual notion of these modules has been proposed by Singh and Srivastava in [11].

The following questions are posed in the papers by Lee and Zhou ([9]), Clark and Huynh ([3]), and Singh and Srivastava ([12]), respectively:

(Q1) Is a simple ring R such that R_R is pseudo-injective right self-injective [3]?

(Q2) Is a simple ring R such that R_R is automorphism-invariant right self-injective [12]?

(Q3) What is the structure of rings whose cyclic right modules are automorphism-invariant [12]?

A module M is called pseudo-injective if, for any submodule A of M, every monomorphism $A \to M$ can be extended to some element of $\mathrm{End}(M)$. Pseudo-injective modules and rings have been discussed by various authors (see, for example [1], [5], [7], [13]). Lee and Zhou showed that a module M is automorphism-invariant if and only if every isomorphism between any two essential submodules of M extends to an automorphism of M [9]. Thus it follows that pseudo-injective modules are automorphism-invariant. Lee and Zhou ask in [9] if the converse holds:

(Q4) Is an automorphism-invariant module pseudo-injective [9]?

In this paper, after proving a useful decomposition theorem for an arbitrary automorphism-invariant module, we show that a prime right nonsingular right automorphism-invariant ring is right self-injective. Using this and the decomposition theorem, we affirmatively answer the questions (Q1), (Q2) and (Q4). Also obtained is a partial answer to (Q3).

For a property P of modules, R is said to have (or be) right P if R_R is a module with P. A closed submodule of a module M is one with no proper essential extensions in M. For submodules A and B of M, B is said to be a complement of A in M if it is maximal among submodules of M trivially intersecting with A. Complement submodules and closed submodules of M coincide, and being a closed submodule is a transitive property. An essential closure of a submodule A of a module M is any closed submodule of M essentially containing A. In a nonsingular module, every submodule has a unique essential closure. A module is called square-free if it does not contain a direct sum of two nonzero isomorphic submodules. Two modules are said to be orthogonal to each other if they do not contain nonzero isomorphic submodules. A module B is said to be A-injective if every homomorphism from any submodule A' of A into B can be extended to an element of $Hom(A, B)$. A detailed treatment of the above concepts and other related facts can be found in [6] and [10]. Throughout the paper, for a module M, $E(M)$ will denote the injective hull of M.

2. A decomposition theorem for automorphism-invariant modules

Before proving our first main result, we will first give some useful lemmas.

Lemma 1. (*[12, Lemma 7]) If M is an automorphism-invariant module with injective hull $E(M) = E_1 \oplus E_2 \oplus E_3$ where $E_1 \cong E_2$, then $M = (M \cap E_1) \oplus (M \cap E_2) \oplus (M \cap E_3)$.*

Lee and Zhou showed in [9] that whenever an automorphism-invariant module M has a decomposition $M = A \oplus B$, A and B are relatively injective. This can be extended as follows:

Lemma 2. *If M is an automorphism-invariant module and A and B are closed submodules of M with $A \cap B = 0$, then A and B are relatively injective. Furthermore, for any monomorphism $h : A \to M$ with $A \cap h(A) = 0$, $h(A)$ is closed in M.*

Proof. First, let K and T be complements of each other in M. Then, $E(M) = E_1 \oplus E_2$, where $E_1 = E(K)$ and $E_2 = E(T)$. Now let $f : E_1 \to E_2$ be any homomorphism. Then the map $g : E(M) \to E(M)$ defined by $g(x_1 + x_2) = x_1 + x_2 + f(x_1)$ ($x_i \in E_i$) is an automorphism, so that $f(K) = (g - 1_{E(M)})(K) \subseteq M$. Hence, $f(K) \subseteq E_2 \cap M = T$. Therefore T is K-injective.

Now if A and B are closed submodules with zero intersection, then, by the above argument, A is injective relative to any complement C of A containing B. Therefore, A is B-injective.

Finally, let $h : A \to M$ be a monomorphism with $h(A) \cap A = 0$, and pick any essential closure K of $h(A)$. Since A is K-injective by the above arguments, $h^{-1} : h(A) \to A$ extends to a monomorphism $t : K \to A$. Therefore, we must have $h(A) = K$.

Theorem 3. *Let M be an automorphism-invariant module. Then the following hold:*

(i) $M = X \oplus Y$ where X is quasi-injective and Y is a square-free module which is orthogonal to X. In this case, X and Y are relatively injective modules.

(ii) If M is nonsingular, then for any two submodules D_1 and D_2 of Y with $D_1 \cap D_2 = 0$, $Hom(D_1, D_2) = 0$.

(iii) If M is nonsingular, $Hom(X, Y) = 0 = Hom(Y, X)$.

Proof. (i) Let $\Gamma = \{(A, B, f) : A, B \leq M, A \cap B = 0$, and $f : A \to B$ is an isomorphism$\}$. Order Γ as follows: $(A, B, f) \leq (A', B', f')$ if $A \subseteq A'$, $B \subseteq B'$, and f' extends f. Then Γ is inductive and there is a maximal element in it, say (A, B, f). Let C' be a complement of $A \oplus B$ in M. C' must be square-free: Otherwise, there would be nonzero submodules X and Y of C' with $X \cap Y = 0$, and an isomorphism $\phi : X \to Y$. But then, $(A \oplus X, B \oplus Y, f \oplus \phi)$ would contradict the maximality of (A, B, f). So C' is square-free. Now define

$g : A \oplus B \oplus C' \to A \oplus B \oplus C'$ via $g(a+b+c) = f^{-1}(b) + f(a) + c$ ($a \in A$, $b \in B$, $c \in C'$). Since M is automorphism-invariant, any isomorphism between two essential submodules of M extends to an automorphism of M, whence g extends to an automorphism g' of M. Let A' be a closed submodule of M essentially containing A. If A were properly contained in A', $g'_{|A'}$ would contradict the maximality mentioned above. Thus, A must be a closed submodule of M. Since closed submodules are preserved under automorphisms, B too is closed in M. Thus, by Lemma 1, $M = (E(A) \cap M) \oplus (E(B) \cap M) \oplus (E(C') \cap M)$. Then, $M = A \oplus B \oplus C'$. Since direct summands of an automorphism-invariant module are again automorphism-invariant, $A \oplus B$ is automorphism-invariant. Now, by Lemma 2, it follows that A and B are relatively injective. Since $A \cong B$, $A \oplus B$ is then quasi-injective. Also, $A \oplus B$ and C' are relatively injective modules. Next, in a similar way to the above argument, one can find a maximal monomorphism $t : B' \to B$ from a submodule $B' \subseteq C'$ into B. Since B is C'-injective, t can be monomorphically extended to a closed submodule of C' essentially containing B'. By the maximality of t, this implies that B' is closed in C'. Also since C' is B-injective, t^{-1} extends monomorphically to an essential closure, say D, of $t(B')$. Since B' would then be essential in the image of D, this implies that $t(B')$ is closed in B. So $t(B')$ is a direct summand of B, since B is quasi-injective. And since B is C'-injective, $t(B')$ is C'-injective, hence B' is a C'-injective submodule of C'. Thus, $C' = B' \oplus C$ for some C. Now, we will show that C and B are orthogonal: Assume that C and B have nonzero isomorphic submodules C_1 and B_1. Then, by square-freeness of C', C_1 and B' are orthogonal modules, and thus, so are B_1 and $t(B')$, so that we would have $B_1 \cap t(B') = 0$. This would contradict the maximality of the monomorphism t. So C and B are orthogonal, whence C and $A \oplus B \oplus B'$ are orthogonal. Furthermore, $A \oplus B \oplus B'$ is quasi-injective. Taking $X = A \oplus B \oplus B'$ and $Y = C$, we obtain the desired conclusion.

(ii) Let $f : D_1 \to D_2$ be a nonzero homomorphism. By the nonsingularity, $Ker(f)$ is closed in D_1 and there is some submodule $L \neq 0$ of D_1 with $Ker(f) \cap L = 0$. But then, $L \cong f(L) \subseteq D_2$, contradicting the square-freeness of Y. Now the conclusion follows.

(iii) Similar to (ii).

Corollary 4. *Any square-full automorphism-invariant module is quasi-injective.*

Remark 5. Before the next result, note that in the proof of Theorem 3 (ii), we have not used the assumption that M is automorphism-invariant, so the statement holds for any nonsingular square-free module.

Recall that a submodule N of a module M is called a fully invariant submodule if, for every endomorphism f of M, $f(N) \subseteq N$.

Theorem 6. *The following hold for a nonsingular square-free module M:*

(i) Every closed submodule of M is a fully invariant submodule of M.

(ii If M is automorphism-invariant, then for any family $\{K_i : i \in I\}$ of closed submodules of M (not necessarily independent), the submodule $\Sigma_{i\in I} K_i$ is automorphism-invariant.

Proof. First, assume that M is square-free and nonsingular. Let K be a closed submodule of M and T be a complement in M of K. Suppose that $f \in \mathrm{End}(M)$ with $f(K) \nsubseteq K$. Let $\pi : E(M) \to E(T)$ be the obvious projection with $Ker(\pi) = E(K)$. Since K is not essential in $f(K) + K$, we have $\pi(K + f(K)) \neq 0$, implying that $\pi(f(K)) \neq 0$, whence $N = T \cap \pi(f(K)) \neq 0$. Then, for $N' = \{x \in K : \pi f(x) \in T\}$, we have $Hom(N', N) \neq 0$, contradicting the assertion preceding this theorem. This proves (i).

Now assume, furthermore, that M is automorphism-invariant, and let $\{K_i : i \in I\}$ be any family of closed submodules of M, and g be an automorphism of $E(\Sigma_{i\in I} K_i)$. Clearly, g can be extended to an automorphism g' of $E(M)$. Since M is automorphism-invariant, we have $g'(M) \subseteq M$. Then, by (i), $g(K_i) = g'(K_i) \subseteq K_i$ for all $i \in I$. This proves (ii).

3. Nonsingular automorphism-invariant rings

In this section we will prove a theorem describing right nonsingular automorphism-invariant rings and answer two questions raised by Singh and Srivastava in [12], and by Clark and Huynh in [3] concerning when an automorphism-invariant or a pseudo-injective ring is self-injective.

Theorem 7. *If R is a right nonsingular right automorphism-invariant ring, then $R \cong S \times T$, where S and T are rings with the following properties:*

(i) S is a right self-injective ring,

(ii) T_T is square-free, and

(iii) Any sum of closed right ideals of T is a two sided-ideal which is automorphism-invariant as a right T-module.

(iv) For any prime ideal P of T which is not essential in T_T, $\frac{T}{P}$ is a division ring.

Proof. By Theorem 3, $R = eR \oplus (1-e)R$ for some idempotent $e \in R$, where eR is quasi-injective, $(1-e)R$ is square-free and

$$Hom(eR, (1-e)R) = 0 = Hom((1-e)R, eR).$$

Hence, $S = eR$ and $T = (1-e)R$ are ideals. Now we have (i) and (ii). Also, (iii) follows from Theorem 6.

We now prove (iv): Let P be a prime ideal of T which is not essential as a right ideal. Take a complement N of P in T_T. If N were not uniform, there would be two nonzero closed right ideals in N, say X and Y with $X \cap Y = 0$. They would then be ideals by the above argument. But this would contradict the

primeness of P. So N is a uniform right ideal of T. Also note that P is a closed submodule of T_T, because if P' is any essential extension of P, we have $P'N = 0$, implying that $P' = P$. So P is closed in T_T, and hence it is a complement in T_T of N. Since $\frac{N \oplus P}{P}$ is essential in $\frac{T}{P}$, this implies that the ring $\frac{T}{P}$ is right uniform. Furthermore, N is a nonsingular uniform automorphism-invariant T-module, so that every nonzero homomorphism between any two submodules is an isomorphism between essential submodules, and thus it extends to an automorphism of N. Therefore, N is a quasi-injective uniform nonsingular $\frac{T}{P}$-module, and thus its endomorphism ring is a division ring. Since $\frac{T}{P}$ essentially contains the nonsingular right ideal $\frac{N \oplus P}{P}$, it is now a prime right uniform and right nonsingular ring (hence a prime right Goldie ring) with the quasi-injective essential right ideal $\frac{N \oplus P}{P}$. But then $\frac{N \oplus P}{P}$ is injective, implying that $P \oplus N = T$. In fact, since $\mathrm{End}_{\frac{T}{P}}(N)$ is a division ring, $\frac{T}{P}$ is a division ring. In particular, N is a simple right ideal and P is a maximal right ideal of T.

Theorem 8. *If R is a prime right non-singular, right automorphism-invariant ring, then R is right self-injective.*

Proof. By Theorem 7 and primeness, it suffices to look at the case when R_R is square-free: If R_R were not uniform, there would be two closed nonzero right ideals A and B with $A \cap B = 0$. But then A and B would be ideals, whence $AB = 0$, contradicting primeness. So R_R is uniform, nonsingular and automorphism invariant. Now it follows, in the same way as in the proof of Theorem 7, that R is right self-injective.

The following example shows that the conclusion of Theorem 8 fails if we take a semiprime ring instead of a prime one.

Example 9. Let $S = \prod_{n \in \mathbb{N}} \mathbb{Z}_2$, and $R = \{(x_n)_{n \in \mathbb{N}}$: all except finitely many x_n are equal to some $a \in \mathbb{Z}_2$ $\}$. Then S is a commutative self-injective ring with $S = E(R_R)$ with only one automorphism, namely the identity. Thus, R is an automorphism-invariant, semiprime nonsingular ring, but it is not self-injective. Teply constructed in [13] the first example of a pseudo-injective module which is not quasi-injective. In fact, the ring R here is a new example of pseudo-injective ring which is not self-injective, by Theorem 16 below.

The following corollary answers the question of Singh and Srivastava in [12].

Corollary 10. *A simple right automorphism-invariant ring is right self-injective.*

The next corollary answers the question raised by Clark and Huynh in [3, Remark 3.4].

Corollary 11. *A simple right pseudo-injective ring is right self-injective.*

4. Rings whose cyclic modules are automorphism-invariant

Characterizing rings via homological properties of their cyclic modules is a problem that has been studied extensively in the last fifty years. A most recent

account of results related to this prototypical problem may be found in [8], and a recent addition in [2]. Another question raised in [12] is the following: What is the structure of rings whose cyclic right modules are automorphism-invariant? The next result addresses this question.

Theorem 12. *Let R be a ring over which every cyclic right R-module is automorphism-invariant. Then $R \cong S \times T$, where S is a semisimple artinian ring, and T is a right square-free ring such that, for any two closed right ideals X and Y of T with $X \cap Y = 0$, $Hom(X, Y) = 0$. In particular, all idempotents of T are central.*

Proof. By the proof of Theorem 3, we have a decomposition $R_R = A \oplus B \oplus B' \oplus C$, where $A \cong B$, B' is isomorphic to a submodule of B, and C is square-free and $A \oplus B \oplus B'$ and C are orthogonal. Let Z be a right ideal in A. Then $\frac{R}{Z} \cong \frac{A}{Z} \oplus B \oplus B' \oplus C$ is automorphism-invariant by assumption. Then, by Lemma 2, $\frac{A}{Z}$ is B-injective, whence A-injective. Similarly, all factors of B, B', and C are A-injective as well.

Now, A is a cyclic projective module all of whose factors are A-injective (and in particular, quasi-injective). So, by [6, Corollary 9.3 (ii)], $A = U_1 \oplus ... \oplus U_n$, where U_i are uniform modules. Take an arbitrary nonzero cyclic submodule U of U_i, for any i. Since U is a sum of factors of A, B, B' and C, it contains a nonzero factor of one of them, call U'. By the above paragraph, U' is A-injective, so it splits in U_i. Thus, $U' = U = U_i$, showing that U_i is simple, whence $A \oplus B \oplus B'$ is semisimple. Since $A \oplus B \oplus B'$ and C are orthogonal projective modules and the former is now semisimple, there are no nonzero homomorphisms between them. Therefore, $A \oplus B \oplus B'$ and C are ideals. So now we have the ring direct sum $R = S \oplus T$ where $S = A \oplus B \oplus B'$ and $T = C$.

Now let X and Y be closed right ideals of T such that $X \cap Y = 0$, and let $f : X \to Y$ be any homomorphism. Set $Y' = f(X)$. This induces an isomorphism $\overline{f} : \frac{X}{K} \to Y'$, where $K = Ker(f)$. It is clear that $\frac{X}{K}$ is a closed submodule of $\frac{T}{K}$. Also, since T_T is square-free, K is essential in X. Choose a complement $\frac{U}{K}$ of $\frac{X}{K} \oplus \frac{Y' \oplus K}{K}$ in $\frac{T}{K}$. Since $\frac{T}{K}$ is automorphism-invariant by assumption and $\frac{X}{K} \cong Y' \cong \frac{Y' \oplus K}{K}$, by the last part of Lemma 2, $\frac{Y' \oplus K}{K}$ is closed in $\frac{T}{K}$. Applying Lemma 1, we obtain $\frac{T}{K} = \frac{X}{K} \oplus \frac{Y' \oplus K}{K} \oplus \frac{U}{K}$. Since $Y' \cap (X + U) \subseteq Y' \cap K = 0$, we have $T = Y' \oplus (X + U)$. So Y'_T is projective, whence the map f above splits. However, since K is essential in X, we have $f = 0$. So, $Hom(X, Y) = 0$. In particular, if $T_T = X \oplus Y$, we have $XY = YX = 0$, whence X and Y are ideals.

Using an alternative argument to the one in the second paragraph of the above proof, we can generalize the decomposition in the theorem as follows:

Proposition 13. *Let M be a module satisfying any one of the following conditions:*

(i) M is cyclic with all factors automorphism-invariant, and generates its cyclic subfactors, or

(ii) M is any automorphism-invariant module whose 2-generated subfactors are automorphism-invariant.

Then $M = X \oplus Y$, where X is semisimple, Y is square-free, and X and Y are orthogonal.

Proof. First note that, by the proof of Theorem 3, we have a decomposition $M = A \oplus B \oplus B' \oplus C$, where $A \cong B$, B' embeds in B, and C is square-free and orthogonal to $A \oplus B \oplus B'$.

(i) In this case, in the same way as in the first paragraph of the proof of Theorem 12, all factors of the modules B ($\cong A$), B' and C are A-injective. Now let A' be any factor of A and D be a cyclic submodule of A'. Since D is generated by M, $D = D_1 + ... + D_n$, where each D_i is a factor of B, B' or C. Since D_1 is A-injective (whence A'-injective), $D_1 \oplus D'_1 = A'$ for some submodule D'_1 of A'. Letting $\pi : D_1 \oplus D'_1 \to D'_1$ be the obvious projection, we have $D = D_1 \oplus (\pi(D_2) + ... + \pi(D_n))$. Each $\pi(D_k)$ again being a factor of B, B' of C, it is A-injective, whence D'_1-injective. By induction on n, we obtain that D is a direct sum of A-injective cyclic modules. Then D is A-injective. Now we have shown that each cyclic subfactor of A is A-injective. By [6, Corollary 7.14], A is semisimple. Therefore, $A \oplus B \oplus B'$ is semisimple, as well. Now set $X = A \oplus B \oplus B'$ and $Y = C$.

(ii) Let $D \subseteq L$ be submodules of A with $\frac{L}{D}$ cyclic, and T be a cyclic submodule of B. By assumption, $\frac{L}{D} \oplus T$ is automorphism-invariant, whence $\frac{L}{D}$ is T-injective. Then, cyclic subfactors of A are B-injective, hence A-injective. Again, by [6, Corollary 7.14], A is semisimple. The conclusion follows in the same way as above.

5. Pseudo-injective modules and automorphism invariant modules coincide

In [9] Lee and Zhou raise the following question: Is an automorphism-invariant module pseudo-injective? In the next theorem, this question is answered affirmatively, also settling [12, Question 2].

First we recall a useful lemma.

Lemma 14. *([6, Lemma 7.5]) Let $M = A \oplus B$. Then A is B-injective if and only if for any submodule C of M with $A \cap C = 0$, there exists some submodule D of M such that $C \subseteq D$ and $A \oplus D = M$.*

Lemma 15. *Assume that $M = A \oplus B$ where, A and B are orthogonal to each other. For any submodule C of M and any monomorphism $f : C \to M$ the following assertions hold:*

(i) $f(C \cap B) \cap B$ is essential in $f(C \cap B)$.

(ii) If B is square-free, then $f(C \cap B) \cap (C \cap B)$ is essential in both $f(C \cap B)$ and $C \cap B$.

Proof. Let C be a submodule of M and $f : C \to M$ be a monomorphism. Assume D is a submodule of $f(C \cap B)$ with $D \cap B = 0$. Then D is embedded (via the obvious projection $A \oplus B \to A$) into A. But D is also isomorphic to a submodule of $C \cap B$. This implies, by orthogonality, that $D = 0$. This proves (i).

Now, assume X is a nonzero submodule of $f(C \cap B)$ with $X \cap (C \cap B) = 0$. Then by (i), $X \cap B \neq 0$, and now $(X \cap B)^2$ embeds in $(X \cap B) \oplus (C \cap B) \subseteq B$, a contradiction to the assumption that B is square-free. Hence, $f(C \cap B) \cap (C \cap B)$ is essential in $f(C \cap B)$. One can see similarly that $f(C \cap B) \cap (C \cap B)$ is also essential in $C \cap B$. This proves (ii).

Theorem 16. *A module M is automorphism-invariant if and only if it is pseudo-injective.*

Proof. The fact that pseudo-injective modules are automorphism-invariant follows from [9]. So, let M be automorphism-invariant, C be a submodule of M, and $f : C \to M$ be a monomorphism. By Theorem 3, $M = A \oplus B$, where A is quasi-injective, B is square-free automorphism-invariant, and A and B are relatively injective. Now let K be a complement in B of $f(C \cap B) \cap (C \cap B)$. Then, by Lemma 15 (ii), $K \oplus [f(C \cap B) \cap (C \cap B)]$ is essential in both $K \oplus (C \cap B)$ (hence in B) and $K \oplus f(C \cap B)$. This implies $[K \oplus f(C \cap B)] \cap A = 0$, and $K \oplus f(C \cap B) \oplus A$ is essential in M.

Since A is B-injective, then by Lemma 14, there exists a submodule B' of M such that $f(C \cap B) \oplus K \subseteq B'$ and $M = A \oplus B'$. In this case, $B' \cong B$. By the above paragraph, $f(C \cap B) \oplus K$ is essential in B'. Since B is automorphism-invariant, the isomorphism $f_{|C \cap B} \oplus 1_K : (C \cap B) \oplus K \to f(C \cap B) \oplus K$ extends to some isomorphism $f' : B \to B'$. So now $f'_{|C \cap B} = f_{|C \cap B}$.

The map $g : C + B \to f(C) + B'$ defined by $g(c + b) = f(c) + f'(b)$ ($c \in C$, $b \in B$) is well-defined and extends f. Now let $\pi : A \oplus B \to A$ be the obvious projection. Then $B + C = B \oplus \pi(C)$. Note that $\pi(C) = (B + C) \cap A$. Since A and B are both A-injective, then M is A-injective, whence $g_{|\pi(C)} : \pi(C) \to M$ extends to some $g' : A \to M$. Clearly, $g'_{|\pi(C)} = g'_{|(B+C) \cap A} = g_{|(B+C) \cap A}$. Now we define $\psi : M \to M$ as follows: For $a \in A, x \in B + C$, $\psi(a + x) = g'(a) + g(x)$. ψ is the desired extension of f to M. Therefore M is pseudo-injective.

Since pseudo-injective modules are known to satisfy the property (C_2) by [5], this also yields the affirmative answer to another question in [12].

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